A new method in diffraction tomography based on the optimization of a topology

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Outline

Introduction: Inverse scattering problem

First method: Shape optimization

Reconstructions

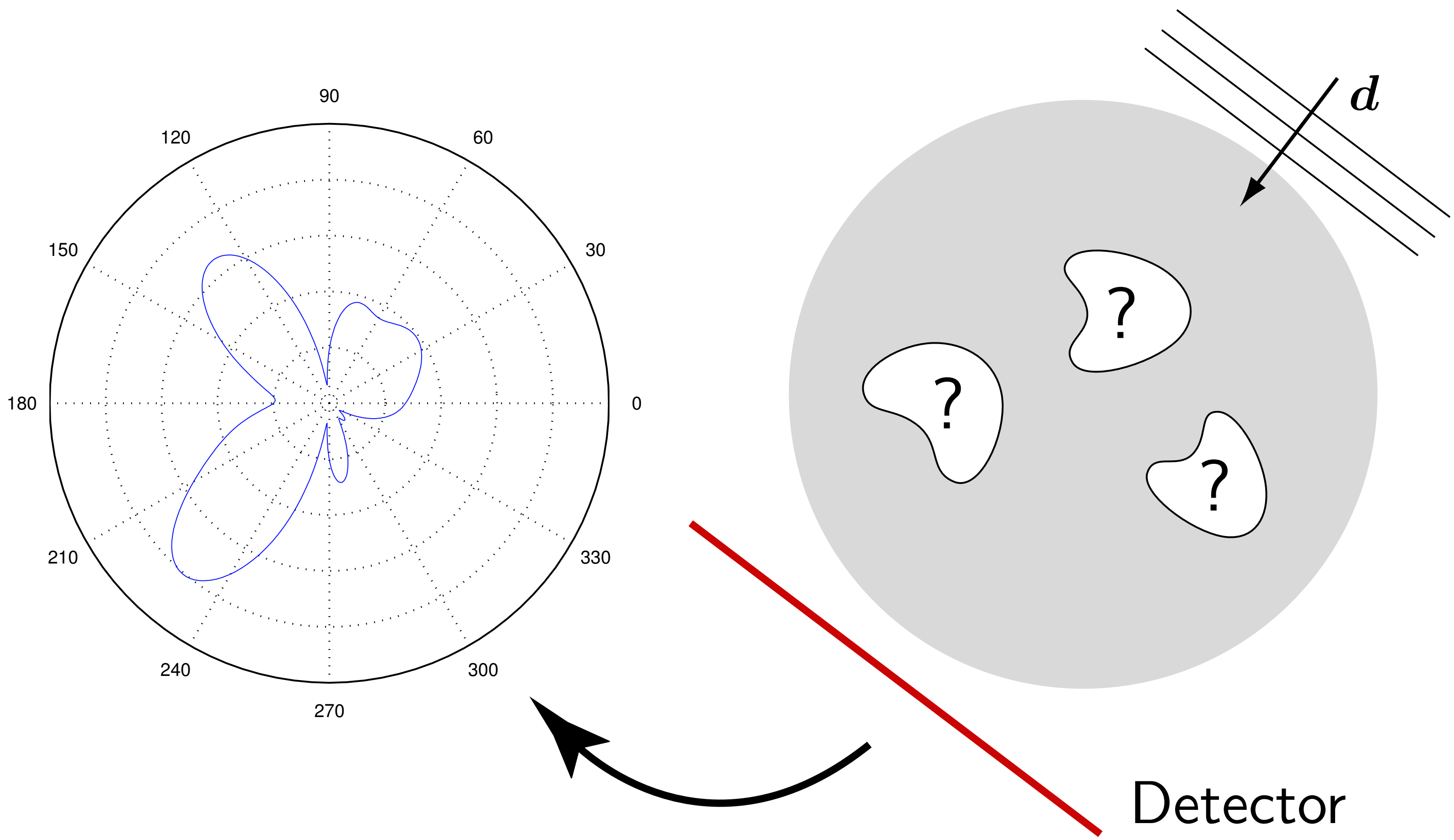
An improved method: Topological derivative

Reconstructions

Conclusions and future work

Introduction: Inverse scattering problem

Reconstruct shape of scatterer(s)



Introduction: Inverse scattering problem

Problem definition: Given

- the measured signal u_m on Γ_s,
- the forward model $a(\Omega; w, u) = \ell(\Omega; w)\ \forall w$,
- the operator $\mathcal{F}(\Omega, u(\Omega)) = u_{\mathrm{sp}}$ on Γ_s,

find the domain Ω such that "$\mathcal{F}(\Omega, u(\Omega)) = u_T$."

Remarks:

- Ω is the domain where the model is valid.
- $a(\Omega; w, u) = \ell(\Omega; w)\ \forall w$ describes the interaction between the incident wave and the medium.

Introduction: Inverse scattering problem

Forward problem: Given $u_{\mathrm{inc}} = \exp(\mathrm{i}k\boldsymbol{x}\cdot\boldsymbol{d})$, find $u = u_s + u_{\mathrm{inc}}$ such that

$$\begin{aligned} -\nabla^2 u - k^2 u &= 0 && \text{in } \mathbb{R}^2, \\ \nabla u \cdot \boldsymbol{n} &= 0 && \text{on the scatterer(s) surface,} \\ \lim_{r\to\infty} r^{1/2}\left(\frac{\partial u_s}{\partial r} - \mathrm{i}k u_s\right) &= 0 && \text{Sommerfeld condition.} \end{aligned}$$

Equivalent weak form: Find u such that

$$a(\Omega; w, u) = \ell(\Omega; w) \qquad \forall w.$$

Introduction: Inverse scattering problem

Solution methods

- Backpropagation algorithm (Devaney, 1982).
  + Fast.
  − Based on the Born or Rytov approximations.

- Nonlinear methods (Chew and Wang, 1990; Kleinman and van den Berg, 1992; Natterer and Wübbeling, 1995).
  + Avoid Born or Rytov approximations.
  − Slow: methods are iterative in nature.

Goal: Construct an **efficient** algorithm that **avoids** these approximations.

Propose two algorithms:

1. Based on the concept of an "optimal shape" for the inverse problem.

2. Improved algorithm based on the "optimal topology" for the inverse problem.

First method: Shape optimization

Iterative method: Find Ω that minimizes

$$j(\Omega) = J(\Omega, u) = \frac{1}{2}\int_{\Gamma_s} \|u - u_m\|^2 \, d\Gamma$$

subject to the constraint

$$a(\Omega; w, u) = \ell(\Omega; w) \quad \forall w.$$

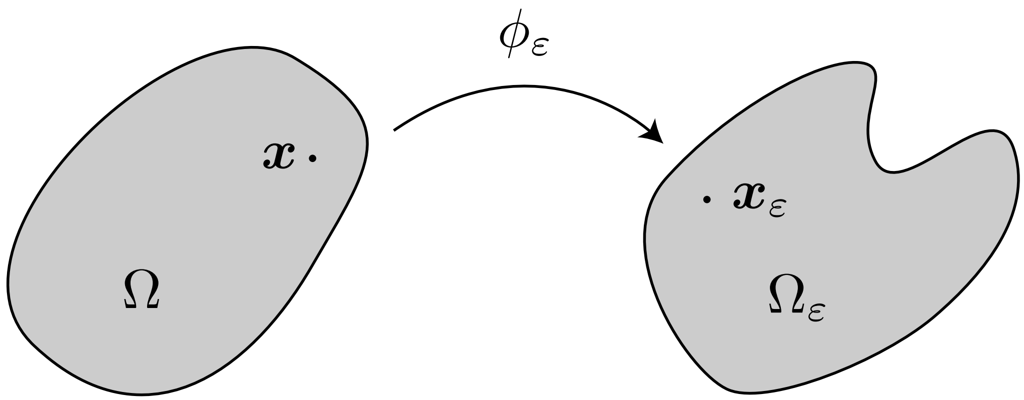
- **Optimization problem** with Ω as the design variable.
- Use **gradient-based** algorithms to solve it.

Need to address the following issues:

1. Differentiation with respect to Ω (**shape differentiation**).
2. How to calculate derivatives in the presence of constraints given by variational equations?

Shape differentiation

Derivative of $j(\Omega)$ in the V-direction



Given the mapping

$$\phi_\varepsilon : \Omega \subset \mathbb{R}^2 \rightarrow \Omega_\varepsilon \subset \mathbb{R}^2$$
$$\phi_\varepsilon(\boldsymbol{x}) = \boldsymbol{x} + \varepsilon \boldsymbol{V}(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \Omega,$$

the **shape derivative** is

$$Dj(\Omega) \cdot \boldsymbol{V} = \frac{d}{d\varepsilon} j(\phi_\varepsilon(\Omega))\bigg|_{\varepsilon=0}.$$

Differentiation in the presence of constraints

We want to calculate

$$Dj(\Omega) \cdot \boldsymbol{V} = \frac{d}{d\varepsilon} J(\Omega_\varepsilon, u_\varepsilon)\Big|_{\varepsilon=0},$$

where u_ε satisfies

$$a(\Omega_\varepsilon; w, u_\varepsilon) = \ell(\Omega_\varepsilon; w) \quad \forall w.$$

Want to avoid computing $\dot{u}_\varepsilon$. For that, introduce the Lagrangian

$$\mathcal{L}(\Omega_\varepsilon, u_\varepsilon, \lambda) = J(\Omega_\varepsilon, u_\varepsilon) + \mathrm{Re}\Big(a(\Omega_\varepsilon; \lambda, u_\varepsilon) - \ell(\Omega_\varepsilon; \lambda)\Big).$$

So

$$\mathcal{L}(\Omega_\varepsilon, u_\varepsilon, \lambda) = J(\Omega_\varepsilon, u_\varepsilon) \quad \forall \lambda.$$

As a consequence,

$$Dj(\Omega) \cdot \boldsymbol{V} = \frac{d}{d\varepsilon} J(\Omega_\varepsilon, u_\varepsilon)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \mathcal{L}(\Omega_\varepsilon, u_\varepsilon, \lambda)\Big|_{\varepsilon=0} \quad \forall \lambda.$$

Seems we did not get much, but...

Differentiation in the presence of constraints

Differentiating the Lagrangian with respect to ε gives

$$\begin{aligned}\frac{d}{d\varepsilon}\mathcal{L}(\Omega_\varepsilon, u_\varepsilon, \lambda) &= D_1 J(\Omega_\varepsilon, u_\varepsilon) + \mathsf{Re}\Big(D_1 a(\Omega_\varepsilon; \lambda, u_\varepsilon)\cdot \boldsymbol{V} - D_1 \ell(\Omega_\varepsilon; \lambda)\cdot \boldsymbol{V}\Big) \\ &\quad + \mathsf{Re}\left(a(\Omega_\varepsilon; \lambda, \dot{u}_\varepsilon)\right) + D_2 J(\Omega_\varepsilon, u_\varepsilon)\cdot \dot{u}_\varepsilon\end{aligned}$$

Select λ that solves the adjoint equation

$$\mathsf{Re}(a(\Omega_\varepsilon; \lambda, w)) + D_2 J(\Omega_\varepsilon, u_\varepsilon)\cdot w = 0 \qquad \forall w.$$

Then

$$\begin{aligned}Dj(\Omega)\cdot \boldsymbol{V} &= \frac{d}{d\varepsilon}\mathcal{L}(\Omega_\varepsilon, u_\varepsilon, \lambda)\Big|_{\varepsilon=0} \\ &= D_1 J(\Omega, u) + \mathsf{Re}\Big(D_1 a(\Omega; \lambda, u)\cdot \boldsymbol{V} - D_1 \ell(\Omega, \lambda)\cdot \boldsymbol{V}\Big) \\ &= G(\Omega, u, \lambda, \boldsymbol{V}).\end{aligned}$$

Differentiation in the presence of constraints

Summary. To compute $Dj(\Omega) \cdot \boldsymbol{V}$

- Solve the forward problem: Find u such that

$$a(\Omega; w, u) = \ell(w) \quad \forall w.$$

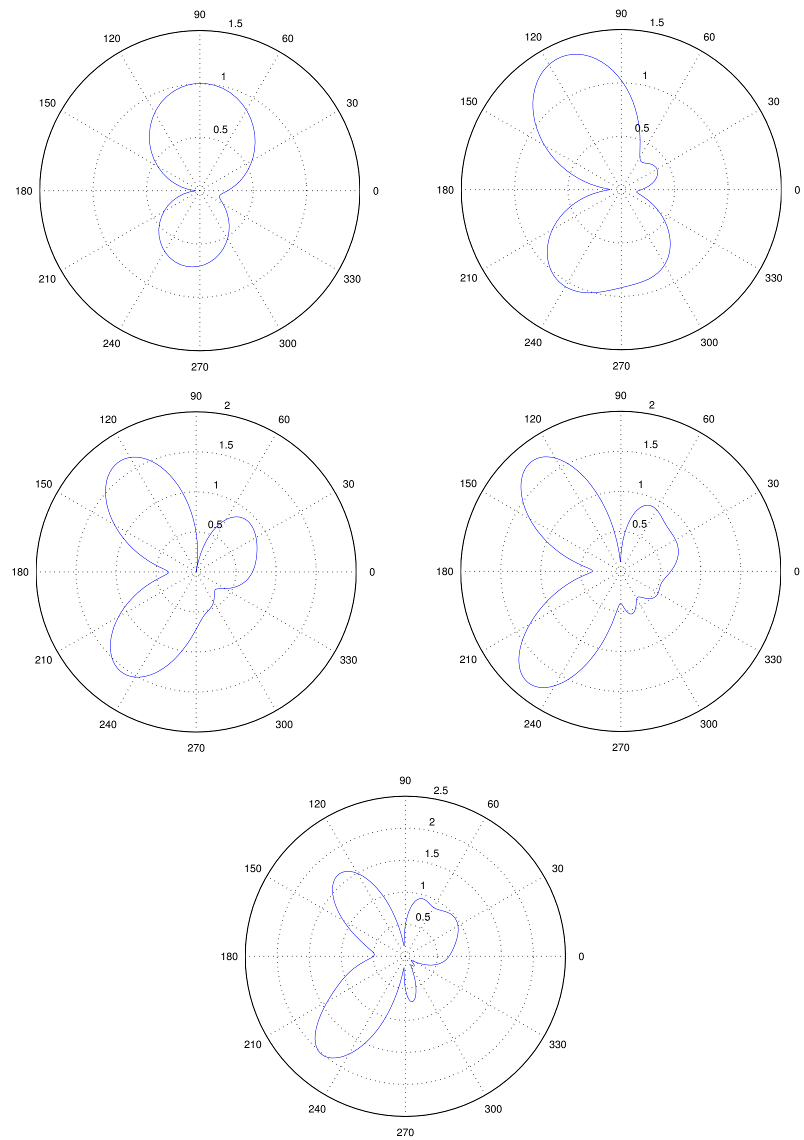
- Solve the adjoint problem: Find λ such that

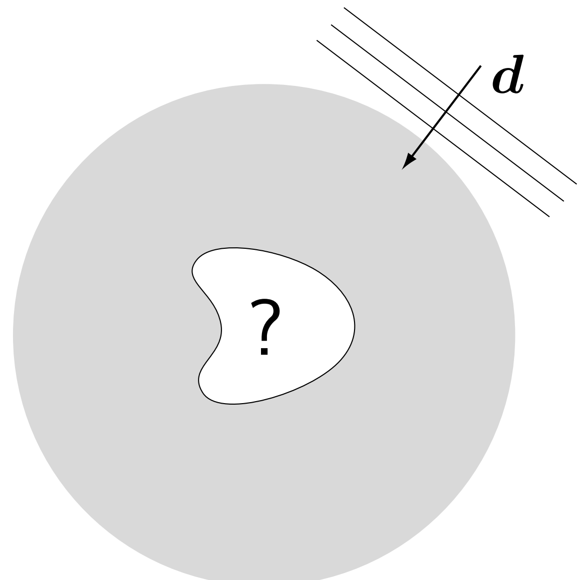
$$a(\Omega; w, \lambda^*) = -(w, (u - u_m)^*)_{\Gamma_s} \quad \forall w.$$

- Compute the shape derivative

$$Dj(\Omega) \cdot \boldsymbol{V} = G(\Omega, u, \lambda, \boldsymbol{V}).$$

Reconstructions using shape optimization





Reconstructions using shape optimization

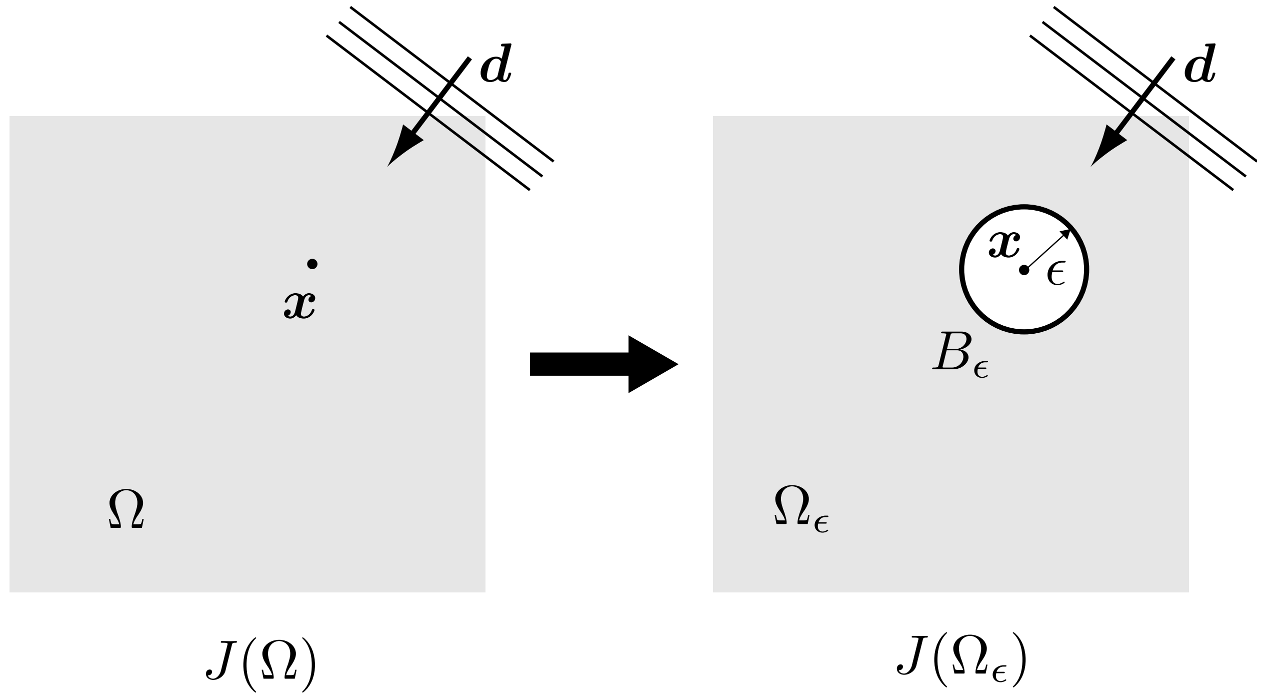
Reconstructions using shape optimization

Search for an improved method

Criticisms to the previous method

- A-priori information (number of scatterers) is needed.
- Robustness problems.
- Method is iterative.

Search for an improved method



What if we could calculate the **scalar field** $D_T(\boldsymbol{x})$ such that

$$J(\Omega_\epsilon) = J(\Omega) + D_T(\boldsymbol{x})f(\epsilon) + o(f(\epsilon))$$

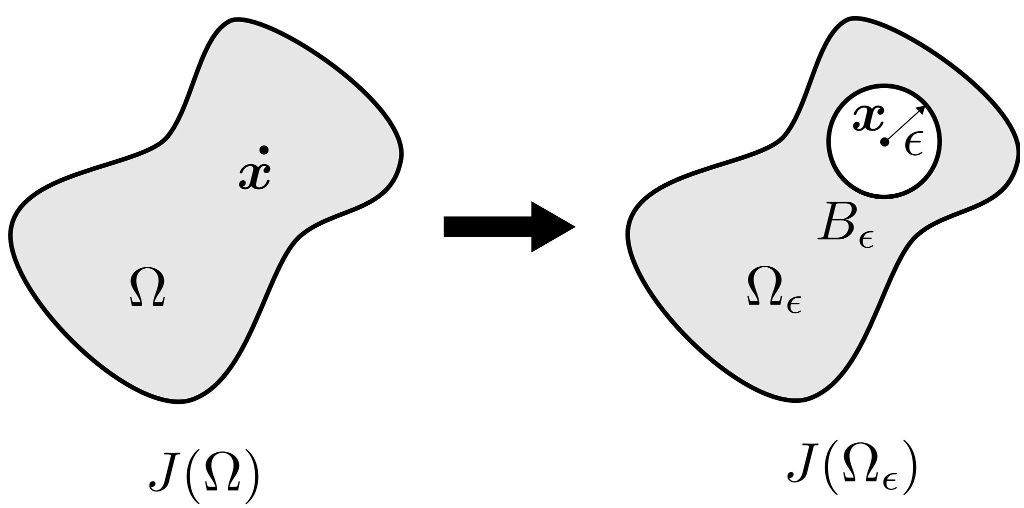
$D_T(\boldsymbol{x})$ can be used as an **indicator** for the position (and shape) of scatterers in the domain Ω.

Search for an improved method

Topological Derivative [Sokolowski, 1999; Masmoudi, 1998]:

$$D_T(\boldsymbol{x}) := \lim_{\epsilon \to 0} \frac{J(\Omega_\epsilon) - J(\Omega)}{f(\epsilon)},$$

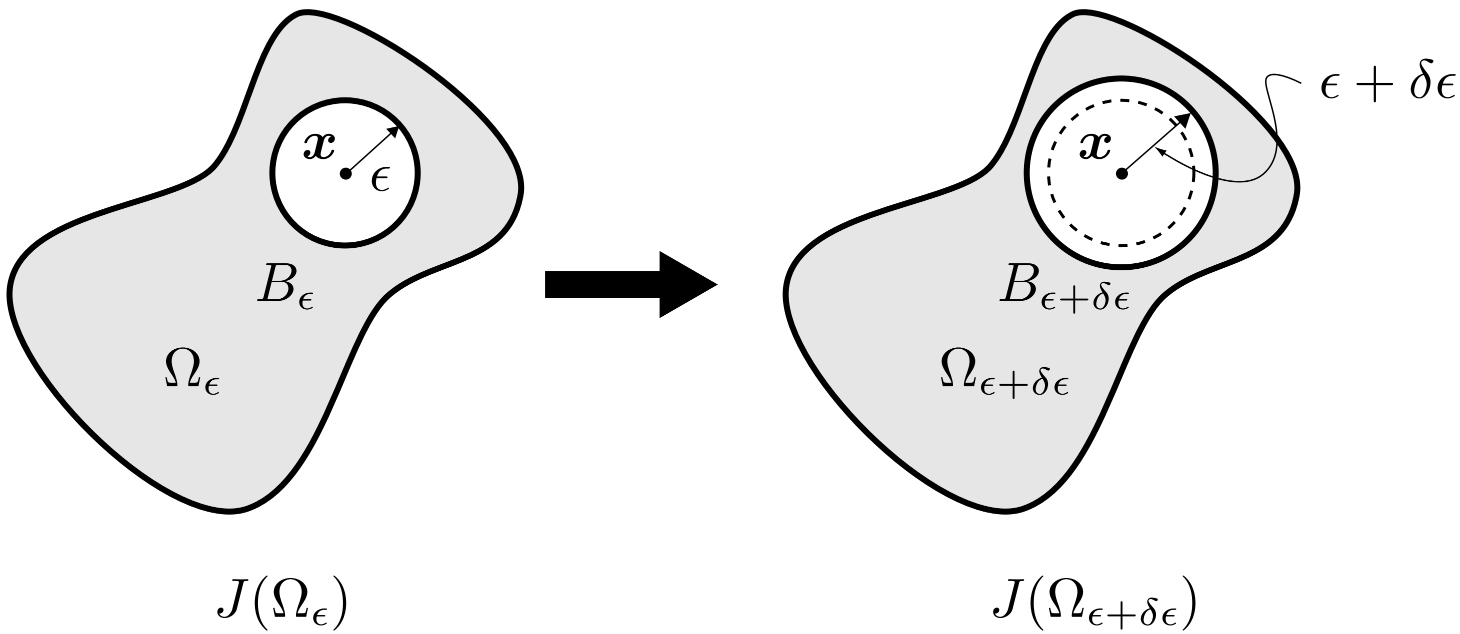
where $\Omega_\epsilon = \Omega \setminus B_\epsilon(\boldsymbol{x})$, $f(\epsilon)$ is a negative function that decreases monotonically and $f(\epsilon) \to 0$ as $\epsilon \to 0^+$.



Search for an improved method

Instead, we can define the topological derivative as follows

$$D_T^1(\boldsymbol{x}) = \lim_{\epsilon \to 0} \left\{ \lim_{\delta\epsilon \to 0} \frac{J(\Omega_{\epsilon+\delta\epsilon}) - J(\Omega_\epsilon)}{f(\epsilon+\delta\epsilon) - f(\epsilon)} \right\}.$$



Search for an improved method

Define:

$$
\begin{aligned}
\Omega_\tau &= \{\boldsymbol{x}_\tau \in \mathbb{R}^n \,|\, \exists \boldsymbol{x} \in \Omega_\epsilon,\, \boldsymbol{x}_\tau = \boldsymbol{x} + \tau \boldsymbol{V}\},\\
\boldsymbol{V} &= \begin{cases} V_n \boldsymbol{n} & V_n < 0 \text{ constant on } \partial B_\epsilon,\\ \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}
\end{aligned}
$$

Theorem:

$$
D_T(\boldsymbol{x}) = D_T^1(\boldsymbol{x}) = \lim_{\epsilon \to 0} \frac{1}{f'(\epsilon)|V_n|} \underbrace{\frac{d}{d\tau} J(\Omega_\tau)\Big|_{\tau=0}}_{(\bullet)}
$$

for $f(\epsilon)$ such that $0 < |D_T(\boldsymbol{x})| < \infty$.

Remark: $(\bullet)$ is the shape derivative!

Second method: Topological derivative

In our case

$$D_T(\boldsymbol{x}) = \mathsf{Re}\left[\nabla\lambda^*(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x}) - k^2\lambda^*(\boldsymbol{x})u(\boldsymbol{x})\right].$$

Both u and λ can be calculated analytically!

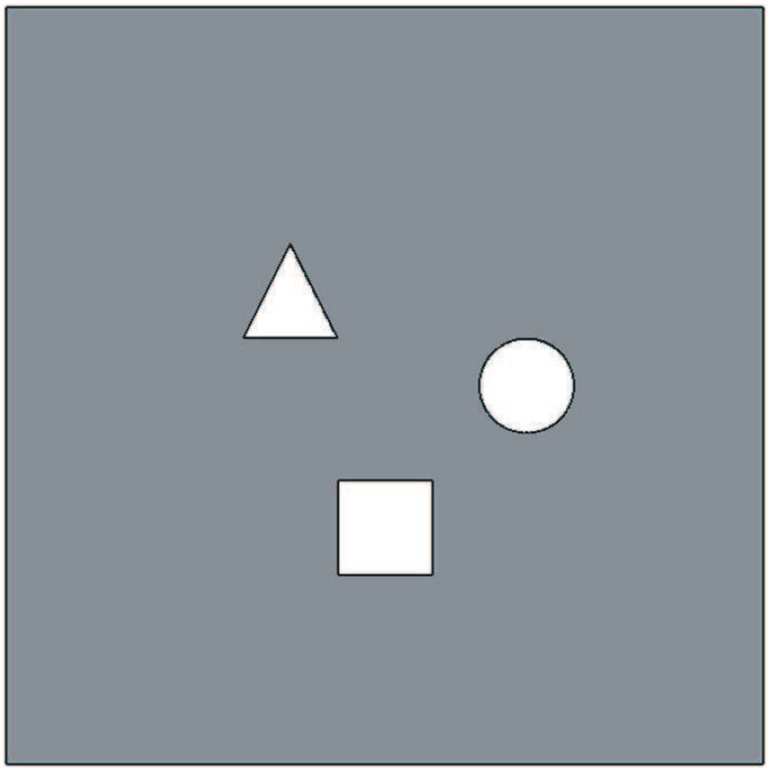
$$\begin{aligned} u(\boldsymbol{x}) &= u_{\mathrm{inc}}(\boldsymbol{x}) = \exp(\mathrm{i}k\boldsymbol{x}\cdot\boldsymbol{d}) \\ \lambda(\boldsymbol{x}(r,\theta)) &= \sum_n A_n J_{|n|}(kr)\exp(\mathrm{i}n\theta) \\ A_n &= f_n(\text{Fourier components of measured signature}) \end{aligned}$$

Method:

Plot $D_T(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$ and look for points where $D_T(\boldsymbol{x})$ attains large values.

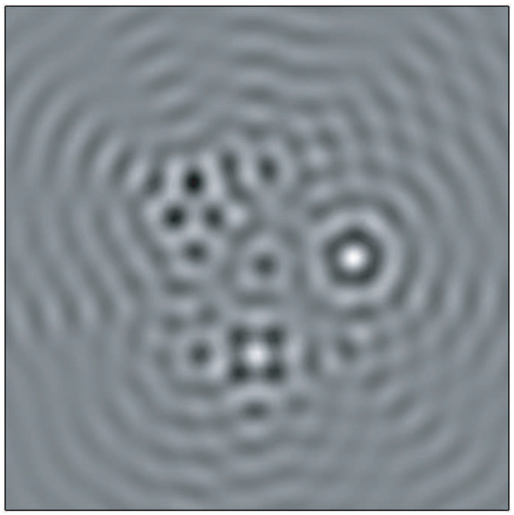
Reconstructions using topological derivative

Target

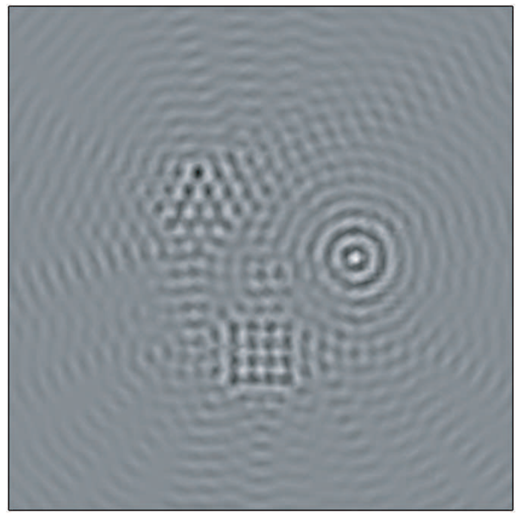


Reconstructions using topological derivative

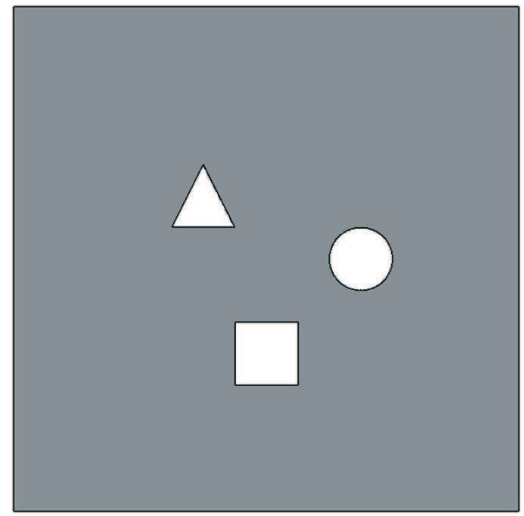
Topological derivatives for $n_{\mathrm{iw}} = n_{\mathrm{dp}} = 120$.



$k = 6$



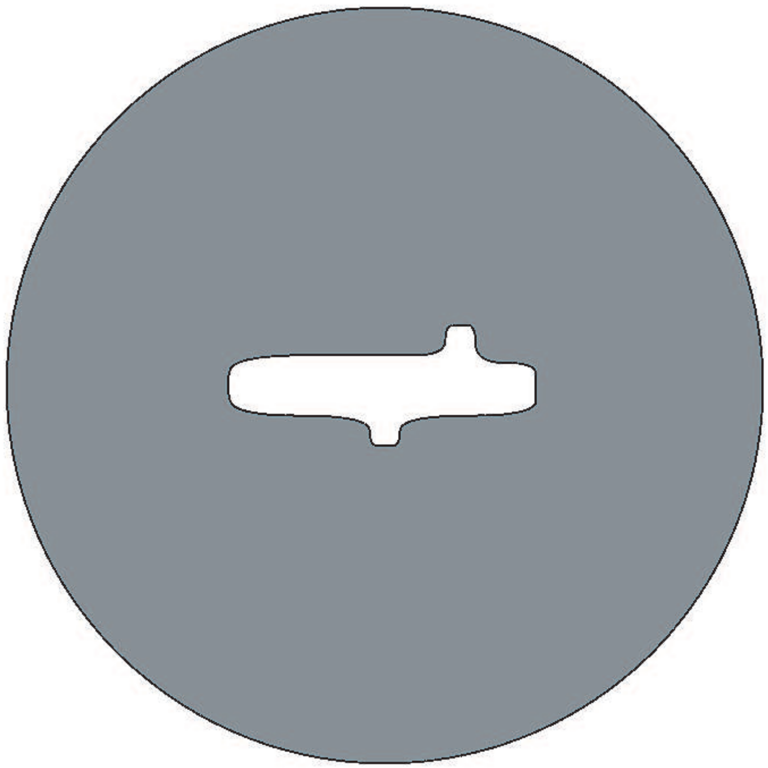
$k = 12$



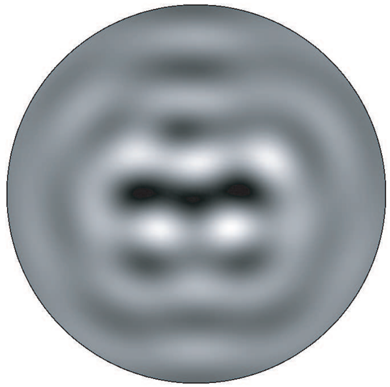
Target

Reconstructions using topological derivative

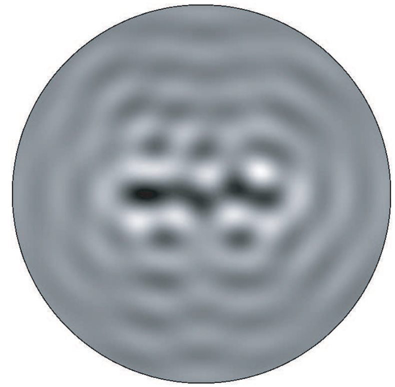
Target



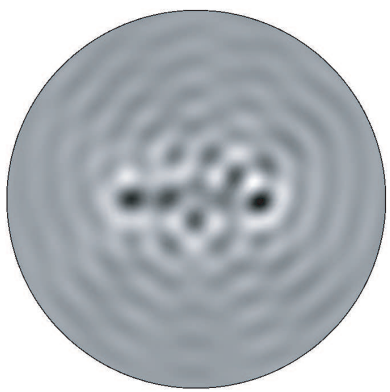
Topological derivatives for $n_{\mathrm{iw}} = n_{\mathrm{dp}} = 120$.



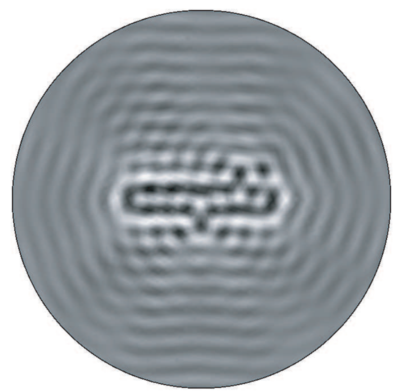
$k = 2$



$k = 3$



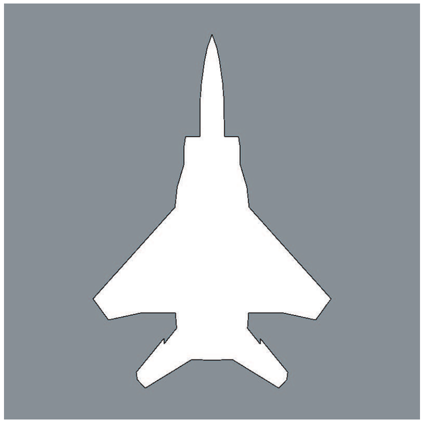
$k = 4$



$k = 6$

Reconstructions using topological derivative

Targets



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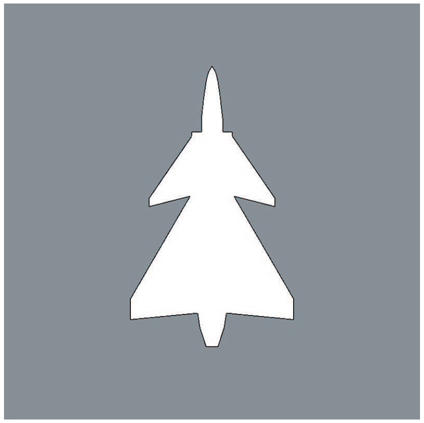

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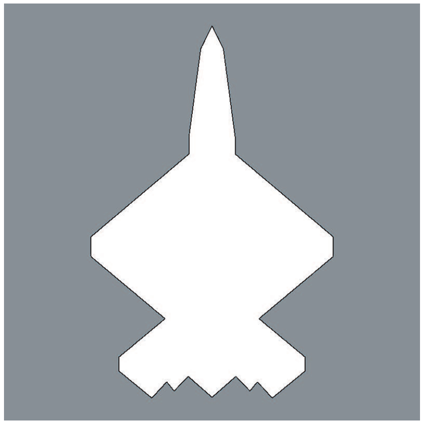

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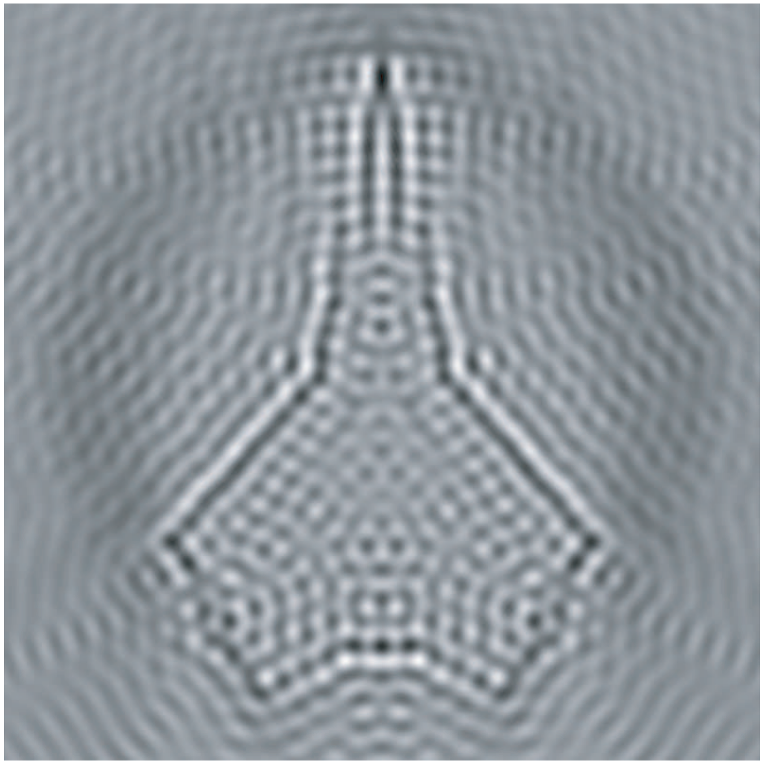
B2

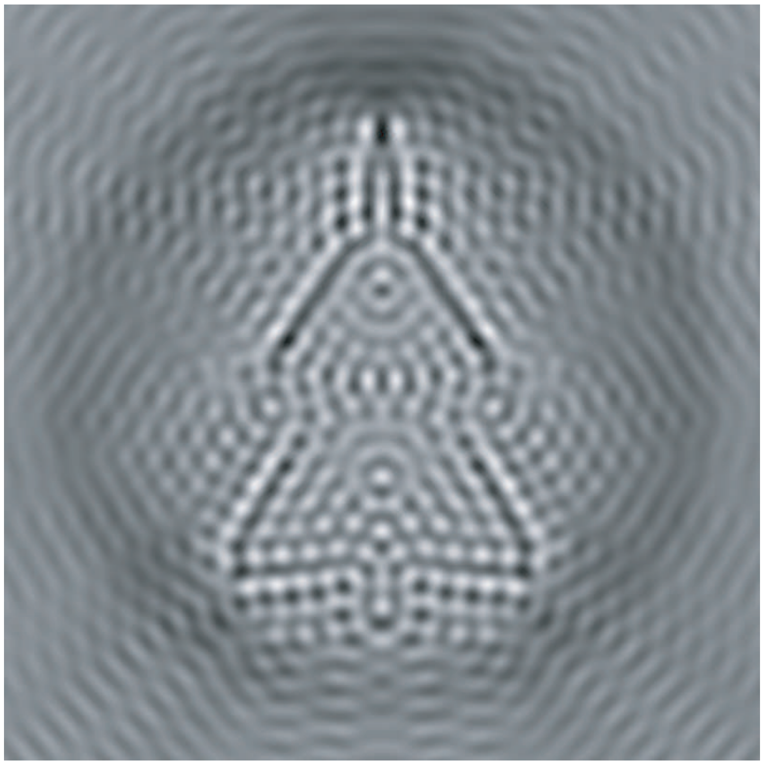
Reconstructions using topological derivative



$\nu = 200\text{MHz}$

Target

Reconstructions using topological derivative



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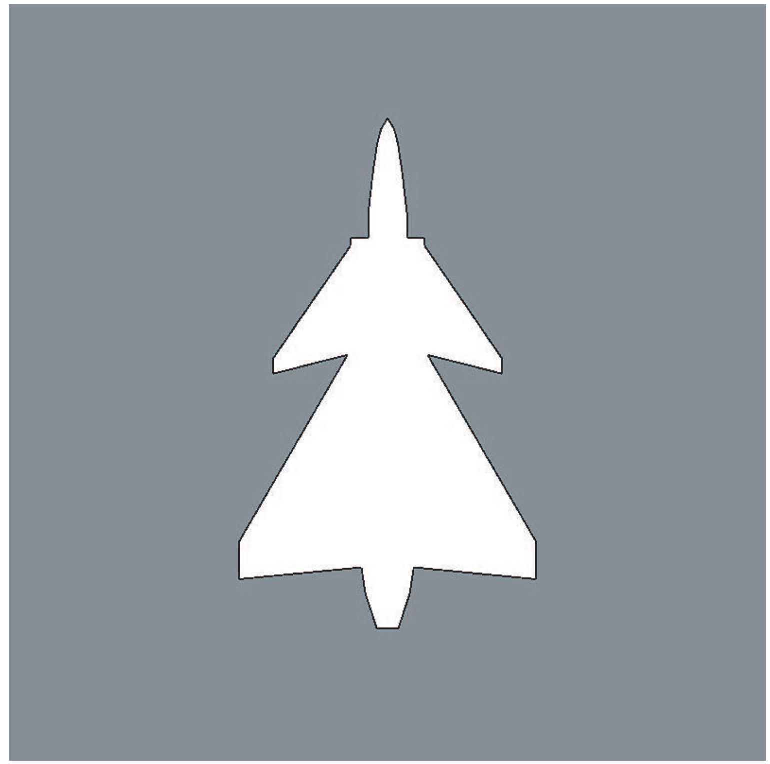

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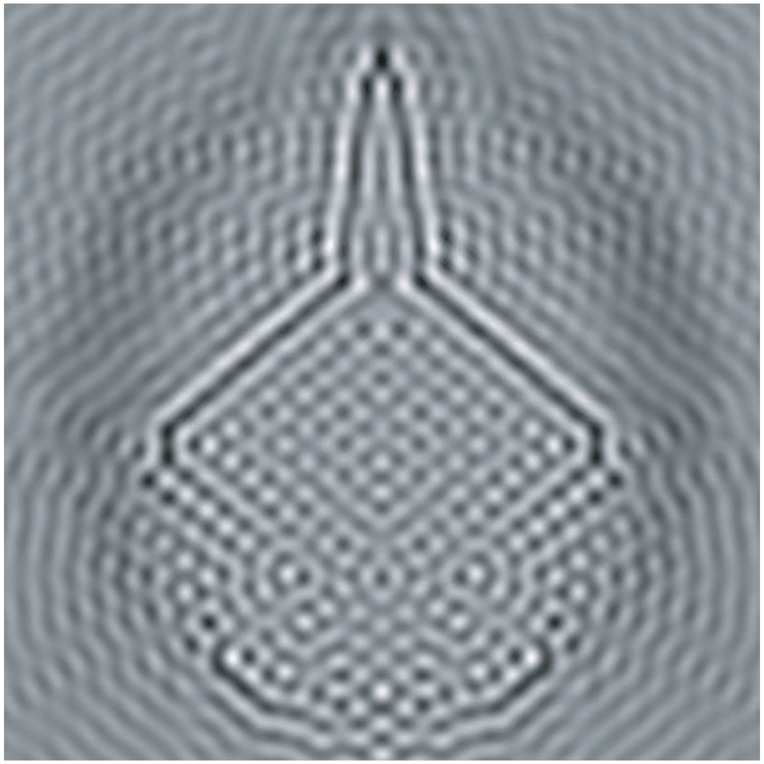
Reconstructions using topological derivative



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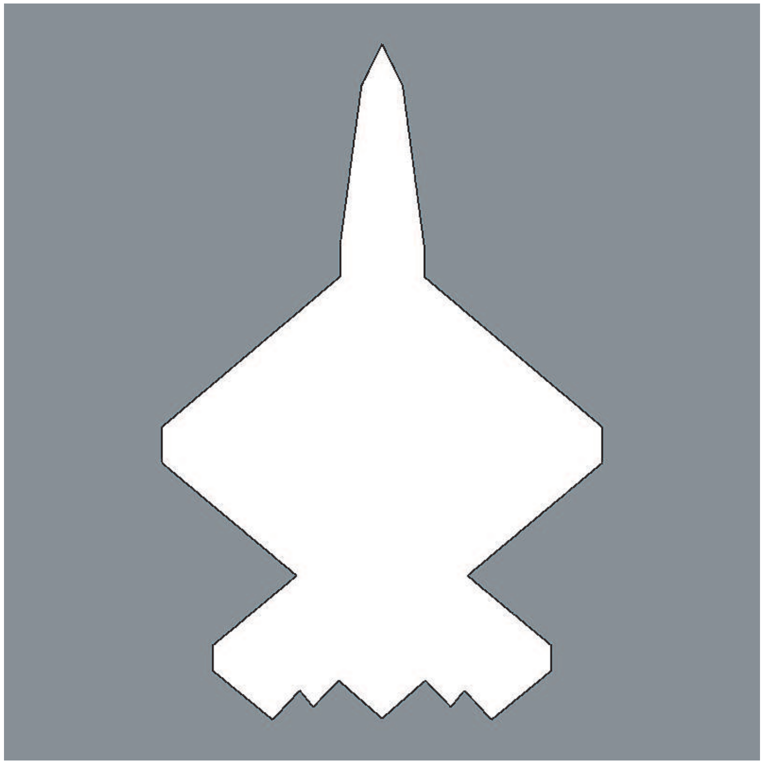

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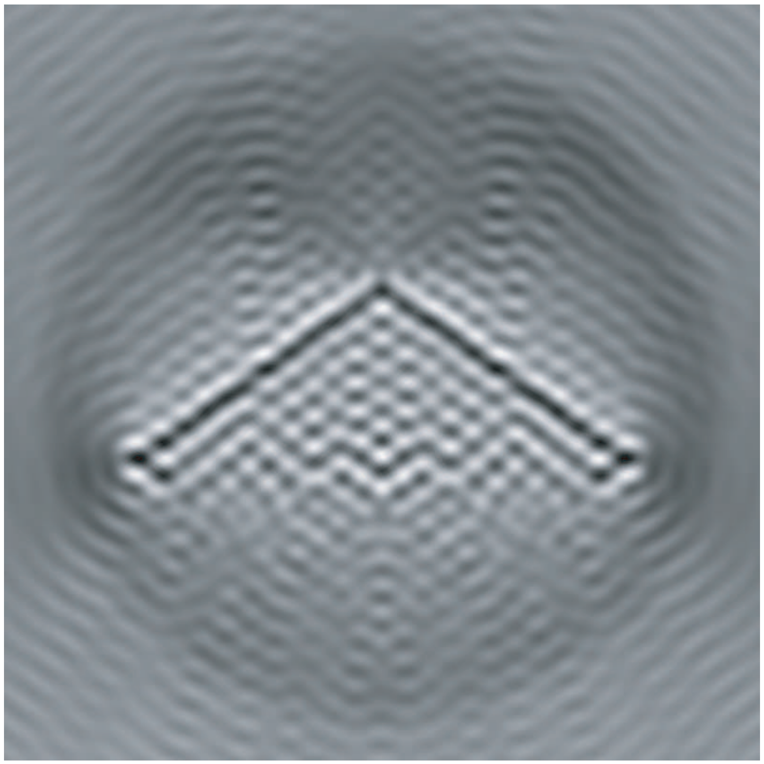
Reconstructions using topological derivative



$\nu = 200\text{MHz}$

Target

Conclusions and future work

Shape sensitivity analysis/topological derivative can be used as a tool to solve inverse scattering problems.

Comparison with other approaches.

3D reconstructions.

Reconstruction of refractive index.

Seismic imaging.

Work with real data!